Approximation of Euler Number Using Gamma Function

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ABSTRACT
This research presents a formula to calculate Euler number using gamma function. The representation is somewhat similar to Taylor series expansion of e. The number e is presented as sum of an integral and decimal part. But it is well known that e is an irrational number, and therefore such an expression for e does not exist in principle, that is why we are approximately representing it for use in computation. The approximation of our formula increases as we increase the value of index in the summation formula. We also analyse the approximation of our formula by both numerical and graphical methods.

KEYWORDS
Euler Number, Gamma Function, Approximation, Computing, L'Hospital's Rule

INTRODUCTION

The number e is a mathematical constant and its value is approximately equal to 2.71828, which is popularly known as Euler number or Napier constant. Its symbol is given in the honour of famous Swiss mathematician Leonhard Euler. The constant is not be confused with Euler-Mascheroni constant or simply Euler constant γ.[1] It is well known that the Euler number e is irrational, that is, it cannot be represented as ratio of integers. Also it is transcendental, that is, it is not a root of any non-zero polynomial with rational coefficients. The number e can be represented as a real number in many ways, as an infinite series, an infinite product, a continued fraction, or a limit of a sequence. The discovery of the constant e is credited to Jacob Bernoulli, who tried to find the value of the given expression $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, an expression that arises in the study of compound interest. It can also be calculated as the sum of the infinite series $\sum_{k=0}^{\infty}\frac{1}{k!}$. The number e has many applications to probability theory, Newton's laws of cooling and in applied science.

In mathematics, the gamma function (represented by the Greek capital letter Γ) is an extension of the factorial function. The gamma function is defined for all complex numbers except the negative integers and zero. For complex numbers with a positive real part, it is defined via a convergent improper integral:

$$\Gamma(n) = \int_{0^+}^{\infty} e^{-x}x^{n-1}dx,\ 0 < \infty < n.$$

If $0 < n < 1$, the integrand becomes infinite as $x \to 0^+$. The integral corresponding to the interval (0,1) is convergent or proper for $0 < n$, while that corresponding to the interval $(1,\infty)$ converges for all n. Hence, $\Gamma(n)$ is well defined integral for all $n > 0$.[2]

The origin of gamma function is due to the problem of extending the factorial to noninteger arguments was apparently first considered by Daniel Bernoulli and Christian Goldbach in the 1720s, and was solved at the end of the same decade by Leonhard Euler. Euler gave two different definitions, first an infinite product of the form $n! = \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^n}{1+\frac{n}{k}}$ and then he wrote to Goldbach again to announce his discovery of the integral representation $n! = \int_0^1 (-\ln s)^n \, ds$ which is valid for $n > 0$. By the change of variable $t = -\ln s$, this integral becomes the familiar Euler integral. Euler published these results in the paper "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt" ("On transcendental progressions, that is, those whose general terms cannot be given algebraically").[3]

Some particular values of the gamma function are given in the following table:

Value of n	Value of $\Gamma(n)$
-1	∞
0	∞
1	1
-3/2	$4\sqrt{\pi}/3$
-1/2	$-2\sqrt{\pi}$
1/2	$\sqrt{\pi}$
3/2	$\sqrt{\pi}/2$
5/2	$3\sqrt{\pi}/4$

The gamma function has caught the interest of some of the most prominent mathematicians of all time. Its history, notably documented by Philip J. Davis in an article that won him the 1963 Chauvenet Prize, reflects many of the major developments within mathematics since the 18th century. In the words of Davis, "each generation has found something of interest to say about the gamma function. Perhaps the next generation will also." [4]

DERIVATION OF EULER NUMBER USING GAMMA FUNCTION

In this section, we derive the Euler number using the notion of gamma function. We restrict the gamma function, $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$, $0 < n < \infty$ only to integral value of n. Replacing n by $(n + 1)$ and setting $x = 1 + a$ in the above equation we get,

$$\Gamma(n+1) = \int_{-1}^{\infty} e^{-(1+a)} (1+a)^n \, da \qquad \textbf{Equation 1.}$$

Now expanding **Equation 1**

$$e \times \Gamma(n+1) = \int_{-1}^{\infty} e^{-a} (1+a)^n \, da$$

$$= \int_{-1}^{0} e^{-a} (1+a)^n \, da + \sum_{i=0}^{n-2} \binom{n}{i} \int_{0}^{\infty} e^{-a} a^i \, da + \binom{n}{n-1} \int_{0}^{\infty} e^{-a} a^{n-1} \, da + \binom{n}{n} \int_{0}^{\infty} e^{-a} a^n \, da,$$

Which on simplification gives

$$e \times \Gamma(n+1) = \int_{-1}^{0} e^{-a} (1+a)^n \, da + \sum_{i=0}^{n-2} \binom{n}{i} \times \Gamma(i+1) + 2 \times \Gamma(n+1). \qquad \textbf{Equation 2.}$$

Consider the function, $\xi(a) = e^{-a} (1+a)^n$ which is defined for all positive integers n. Now using Simpson's rule $\int_{\alpha}^{\beta} f(x) \approx \frac{\beta - \alpha}{6} \times \left(f(\alpha) + f(\beta) + 4 \times \left(\frac{\alpha + \beta}{2}\right)\right)$ [5] of numerical integration for approximation of definite integral of $\xi(a)$ from -1 to 0, we obtain

$$\int_{-1}^{0} e^{-a} (1+a)^n \, da \approx \left(\frac{0-(-1)}{6}\right) \times \left(f(-1) + f(0) + 4 \times f(-1/2)\right)$$

$$\approx \frac{1}{6} \times \left(0 + 1 + \frac{4 \times e^{1/2}}{2^n}\right) \approx \frac{1}{6} + \frac{e^{1/2}}{6 \times 2^{n-2}}$$

which is approximately $\frac{0.2}{2^{n-2}}$. For larger values of n, the quantity $\frac{0.2}{2^{n-2}}$ will approach to zero. But for better approximation, the whole integral of the function $\xi(a)$ will be zero, as the quantity 0.1 is minute and adding it does not affect the integral. Thus, we have $\int_{-1}^{0} e^{-a} (1+a)^n \, da \approx 0$

Therefore **Equation 2** reduces to

$$e \times \Gamma(n+1) \approx \sum_{i=0}^{n-2} \binom{n}{i} \times \Gamma(i+1) + 2 \times \Gamma(n+1).$$

Using the property $\Gamma(t+1) = t!$[2], we get

$$e \times n! \approx \sum_{i=0}^{n-2} \binom{n}{i} \times i! + 2 \times n!,$$

which on simplification gives

$$e \approx \sum_{i=0}^{n-2} \frac{1}{(n-i)!} + 2, \text{ where } n \geq 2. \qquad \textbf{Equation 3.}$$

The following table represents the values of the Euler number e for different values of n using **Equation 3.**

Value of n	Value of e
2	2.5
3	2. 6666666666666667
4	2.7083333333333333
5	2.7166666666666667
6	2.7180555555555556
7	2.7182539682539683
8	2.7182787698412698
9	2.7182815255731922
10	2.7182818011463845
11	2.7182818261984929
12	2.7182818282861686
13	2.718281828446759
14	2. 7182818284582297
15	2.7182818284589945
16	2.7182818284590423
17	2.7182818284590451
18	2.7182818284590452
19	2.7182818284590452
20	2.7182818284590452

ANALYSIS OF APPROXIMATION

In this section, the approximation of the Euler number e given in **Equation 3** is analysed numerically and graphically.

1. Numerical Method

It is observed that by substituting $x = 1 + a$ in **Equation 1**, the limit changed from -1 to ∞, but the actual upper limit was $\infty - 1$. Also we approximated the integral of $\xi(a)$ from -1 to 0 to zero. This brings an approximation in **Equation 3**.

The next graphical approach emphasizes the cause of approximation more clearly.

2. Graphical Method

Differentiating the function $\xi(a) = e^{-a}(1+a)^n$ with respect to a and computing $\xi(a)$ at -1 and 0, we get

$$\xi(-1) = 0,\ \xi(0) = 1,\ \frac{d\,\xi(a)}{da} = e^{-a}(1+a)^{n-1}(n-(a+1))$$

To compute the limit of $\xi(a)$ when $a \to \infty$, we expand the expression $(1 + a)^n$ by binomial theorem and evaluate the limit of each term using L'Hospital's rule, since each term (excluding first term) is in $\frac{\infty}{\infty}$ form. Further, differentiating each term (excluding first term) to numbers of times the respective power of 'a', as e^a never vanishes, all terms will be in $\frac{1}{\infty}$ form. Therefore we get

$$\lim_{a\to\infty} \frac{\left(\sum_{k=0}^{n} \binom{n}{k} a^k \right)}{e^a} = 0$$

Equation 4.

Now we observe the graphs of $\int_{-1}^{\infty} e^{-a} (1 + a)^n \, da$ for different values of n

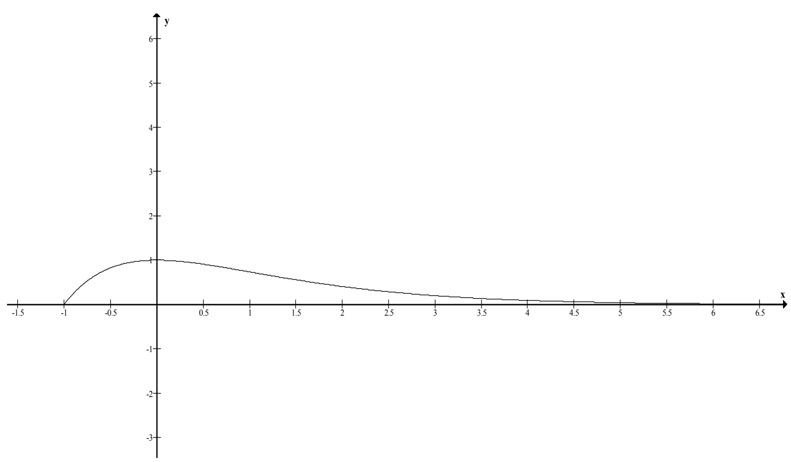

Figure 1. $n = 1$

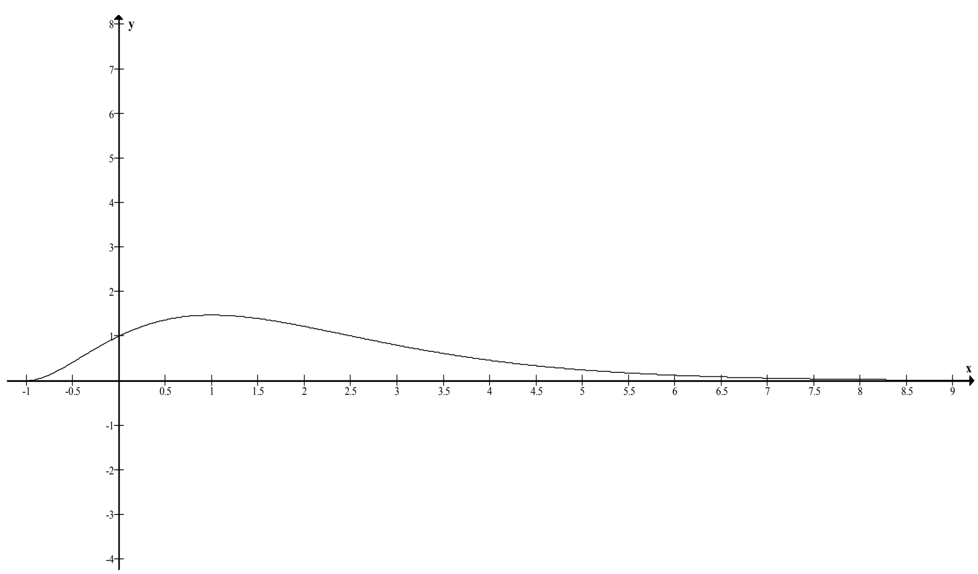

Figure 2. $n = 2$

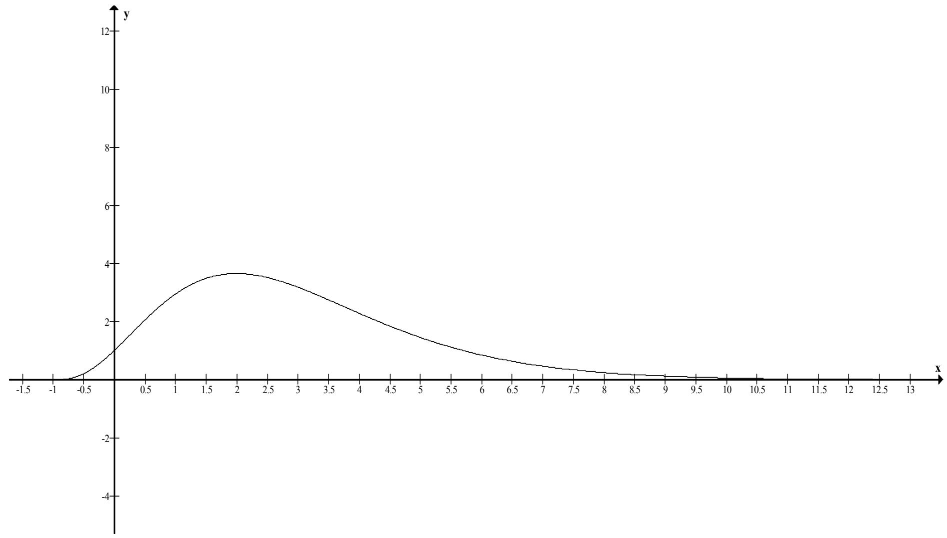

Figure 3. $n = 3$

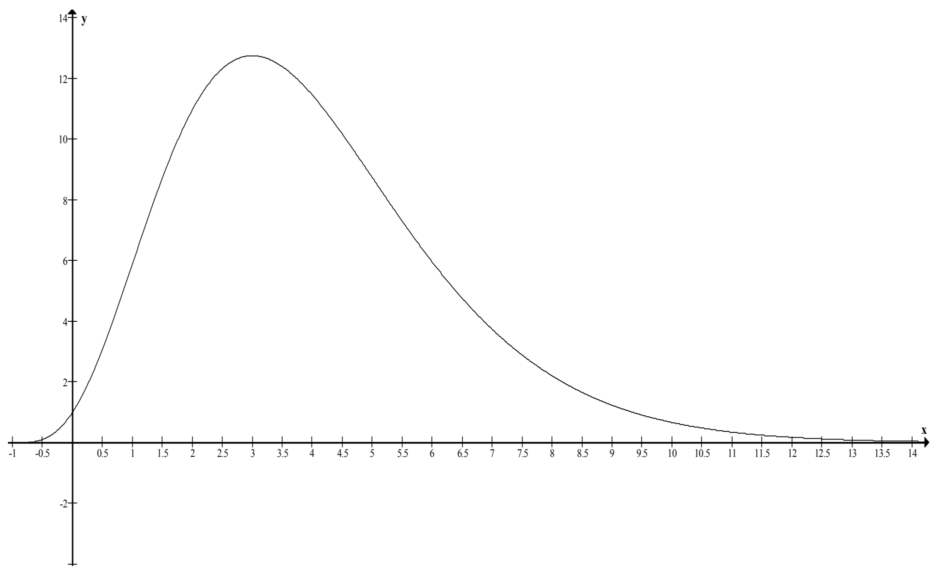

Figure 4. $n = 4$

ANALYSIS OF GRAPHS

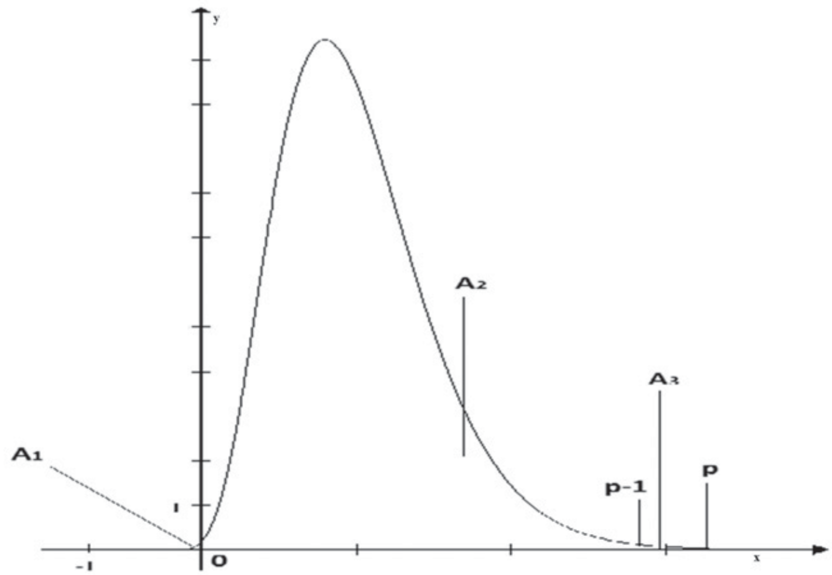

Figure 5. Rough graph of **Figure 1, Figure 2, Figure 3** and **Figure 4**.

Analysing **Figure 5**, we observe that as the value of n increases the steepness of graph between -1 and 0 towards origin increases which means the graph is more approaching towards origin. Also the concave graph between 0 to ∞ is going more upward. In the above figure, we observe that the area $A_{Total}=A_1+A_2$, where A_1 and A_2 are area of regions as shown in the above **Figure 5**.

Using Simpson rule for approximating the integrals and using **Equation 4**, we obtain

$$\int_{p-1}^{p} e^{-a}(1+a)^n da \approx \left(\frac{p-(p-1)}{6}\right) \times \left(f(p-1) + f(p) + 4 \times f\left(\frac{p-1+p}{2}\right)\right) \approx \frac{1}{6} \times (f(p) + f(p) + 4 \times f(p)).$$

As $p \to \infty$, $p - 1 \to \infty$ and also $2p-1 \to \infty$. Therefore, we have

$$\int_{p-1}^{p} e^{-a}\ (1+a)^n\ da \approx f(p) \approx \lim_{p \to \infty} \frac{(1+p)^n}{e^p} \approx 0.$$

The integral $A_3 = \int_{p-1}^{p} e^{-a}(1+a)^n da$ and the integral of $\xi(a)$ from -1 to 0, A_1 , both are tending to same value zero, therefore we must have $A_1 \approx A_3$. Hence replacing area A_1 by area A_3, we have $A_{Total} \approx A_2 + A_3$.

CONCLUSION

Though the Euler number is evaluated using Taylor series, in this study, we have given an alternative method of it to approximate the Euler number using gamma function. We have also connected the graphs with the mathematical equation and observe how the curve varies to fulfil the approximation process.

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PRESS SUMMARY

In this paper, we developed a formula that gives an approximate value of the Euler number e using gamma function. Its expansion is the sum of integer part and the fractional part. Also, the expansion of series can be terminated as per our preference by choosing favourable index in the summation formula. We have approximated the Euler number e in the given formula for a specific value; however, there is a scope to extend the formula for approximating e in a new way using asymptotic notation.