# Construction of Higher Orthogonal Polynomials Through a New Inner Product, $\langle\cdot,\cdot\rangle_p$ In a Countable Real $L^p$-space

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ABSTRACT

This research work places a new and consistent inner product $\langle\cdot,\cdot\rangle_p$ on a countable family of the real $L^p$ function spaces, proves generalizations of some of the inequalities of the classical inner product for $\langle\cdot,\cdot\rangle_p$ provides a construction of a specie of Higher Orthogonal Polynomials in these inner-product–admissible function spaces, and ultimately brings us to a study of the Generalized Fourier Series Expansion in terms of these polynomials. First, the reputation of this new inner product is established by the proofs of various inequalities and identities, all of which are found to be generalizations of the classical inequalities of functional analysis. Thereafter two orthogonalities of $\langle\cdot,\cdot\rangle_p$ (which coincide at p = 2) are defined while the Gram-Schmidt orthonormalization procedure is considered and lifted to accommodate this product, out of which emerges a set of higher orthogonal polynomials in $L^p[-1,1]$ that reduce to the Legendre Polynomials at $p = 2$. We argue that this inner product provides a formidable tool for the investigation of Harmonic Analysis on the real $L^p$ function spaces for $p$ other than $p = 2$, and a revisit of the various fields where the theory of inner product spaces is indispensable is recommended for further studies.

## I. INTRODUCTION

In a normed linear space we can add vectors and multiply them by scalars as in elementary vector algebra. Furthermore the norm, $\| \|\cdot\| \|$, on such space generalizes the elementary concept of the length of a vector. However what is missing in a general normed space which could be introduced is an analogue of the familiar dot product, *i.e.* $a\cdot b=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3$ for $\underline{a}=(\alpha_i)$, $\underline{b}=(\beta_i)$, $i$ = 1,2,3, and the resulting formulae, notably $|\underline{a}|=\sqrt{\underline{a}\cdot\underline{a}}$ and the condition for orthogonality, $\underline{a}\cdot\underline{b}=0$, which are important tools in many applications. Hence the possibility of generalizing the dot product and orthogonality to arbitrary vector spaces should be of interest. In actual fact this consideration was done and led to the discovery of the so-called Hilbert space [1], named for David Hilbert (1862-1943) whose 1912 paper on integral equations inaugurated this vast theory of abstract space [2, 3].

In the axiomatic definition of a Hilbert space given much later by J. von Neumann (1927) [4] and refined by mathematicians like H. Löwig (1934) [5], F. Rellich (1934) [6] and F. Riesz (1934) [7] an arbitrary vector space $\mathrm{X}$ was considered and on it a mapping $\langle\cdot,\cdot\rangle$, defined on $\mathrm{X}$ x $\mathrm{X}$ into the scalar field K of $\mathrm{X}$ with the properties that for all f,g,h $\in$ $\mathrm{X}$ and $\alpha$ $\in \mathrm{X}$,

(i) $\langle f+g,h\rangle=\langle f,h\rangle+\langle g,h\rangle$

(ii) $\langle \alpha f,g\rangle=\alpha\cdot\langle f,g\rangle$

(iii) $\langle f,g\rangle=\overline{\langle g,f\rangle}$

(iv) $\langle f,f\rangle\geq 0$ and $\langle f,f\rangle=0$ if and only if $f=o$

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This mapping, called an inner product, was then used to define a norm, $\| \|\cdot\| \|$, on $\mathrm{X}$ as $\|f\| = \sqrt{\langle f, f\rangle}$ and a metric as $d(f,g) = \sqrt{\langle f-g, f-g\rangle}$. Thus was laid the foundation of a space, which was to generalize the Euclidean space, unite various other spaces and was to prove valuable in practical pursuits in the theory of Quantum Mechanics, Integral Equations, Approximation Theory, etc.

However, it cam about that when the inner product $l^2$-space was introduced in 1912 it was found that the equivalent inner product for the Lebesgue integrable functions only generated the $L^2$-norm, $\| \|\cdot\| \|_2$, thus making only $L^2$ an inner product space out of all the $L^p$ function spaces. These are provable facts in functional analysis that are not contested. In what follows, we revisit the present definition of an inner product, then modify it and seek the necessary and sufficient conditions for the $L^p$ function spaces and for all $p \in 2\mathrm{IN}$.

To start with let us assume the existence of an inner product on some, if not all, of the real $L^p$ function spaces and let us denote it by $\langle\cdot,\cdot\rangle_p$. From undergraduate knowledge of the classical definition of an inner product the following axioms must be satisfied by $\langle\cdot,\cdot\rangle_p$ if this inner product is to justify its existence:

$f, g, h \in L^p$, $\alpha \in R$ and some $p \in (1, 00)$, (if not all), we must have

(i) $\langle f+g, h\rangle_p = \langle f, h\rangle_p + \langle g, h\rangle_p$

(ii) $\langle \alpha f, g\rangle_p = \alpha \cdot \langle f, g\rangle_p$

(iii) $\langle f, f\rangle_p \geq 0$ and $\langle f, f\rangle_p = 0$ if, and only if, $f = 0$

(iv) $\langle f, g\rangle_p = \langle g, f\rangle_{p*}$

where the meaning of $p*$ is explained when the structure of the inner product is exhibited below. In order to therefore test the truth of the above inner-product axioms it would be necessary that one writes out the structure of $\langle\cdot,\cdot\rangle_p$ in terms of the classical inner product, $\langle\cdot,\cdot\rangle$. To this end we define the new inner product as

$$\langle f, g\rangle_p = \langle f, g^{p-1}\rangle, \text{ for } f, g \in L^p,$$

and test the above axioms for the admissible values of $p$, which will be found later. This is shown below by using the definition of $\langle\cdot,\cdot\rangle_p$ and the properties of $\langle\cdot,\cdot\rangle$,

(i)

$$\langle f+g, h\rangle_p = \langle f+g, h^{p-1}\rangle$$

$$= \langle f, h^{p-1}\rangle + \langle g, h^{p-1}\rangle$$
$$= \langle f, h\rangle_p + \langle g, h\rangle_p$$

(ii) $\langle \alpha f, g\rangle_p = \langle \alpha f, g^{p-1}\rangle$

$$= \alpha \cdot \langle f, g^{p-1}\rangle + \alpha \cdot \langle f, g\rangle_p$$

(iii) $\langle f, f\rangle_p = \langle f, f^{p-1}\rangle \geq 0,$ and
$\langle f, f\rangle_p = \langle f, f^{p-1}\rangle = 0$
if and only if, $f = 0$

(iv) $\langle f, g\rangle_p = \langle f, g^{p-1}\rangle = \langle g^{p-1}, f\rangle$
$= \langle g, f\rangle_{p*}$,

where $p*$ in (iv) means that the introduced power *(p-1)* now goes to the first entry in the inner product.

It follows immediately that $\langle\cdot,\cdot\rangle_p$, as defined above in terms of the classical inner product $\langle\cdot,\cdot\rangle$, is indeed an inner product on $L^p$ which makes allowance for the inclusion of *p* and for its variation. This new structure, which is called an $L^p$ inner product, will obviously induce more than just the $L^2$-norm, $\|\cdot\|_2$, and generalize the outlook on $L^2$. The consideration of the values of *p* for which $\langle\cdot,\cdot\rangle_p$ induces a consistent $L^p$-norm is postponed to a later section after we have established some of its basic properties, which will also help in the proof of the axioms of a norm.

However, in order to give a precise definition of an inner product $L^p$ function space, the author wishes to anticipate section VI (infra) of the paper by stating that $\langle\cdot,\cdot\rangle_p$ induces a consistent $L^p$-norm only for all $p\in 2\,I\!N$. This brings us to the following definition.

## II. DEFINITIONS

An inner product $L^p$ function space is a pair $\left(L^p,\langle\cdot,\cdot\rangle_p\right)$ where $L^p$ is a real linear function space and $\langle\cdot,\cdot\rangle_p$ satisfies the following axioms for $f,g,h\in L^p,\alpha\in R$ and all $p\in 2\,I\!N$;

(i) $\langle f+g,h\rangle_p=\langle f,h\rangle_p+\langle g,h\rangle_p$

(ii) $\langle \alpha f,g\rangle_p=\alpha\cdot\langle f,g\rangle_p$

(iii) $\langle f,g\rangle_p=\langle g,f\rangle_{p^*}$

(iv) $\langle f,f\rangle_p\geq 0$ and $\langle f,f\rangle_p=0$ if, and only if, $f=0$.

The following gives some of the properties of $\langle\cdot,\cdot\rangle_p$ all of which generalize those of $\langle\cdot,\cdot\rangle_2=\langle\cdot,\cdot\rangle$.

## III. IDENTITIES

Let $f,g,h\in L^p$ and $\alpha,\beta\in R$, then

(i) $\langle \alpha f+\beta g,h\rangle_p=\alpha\cdot\langle f,h\rangle_p+\beta\cdot\langle g,h\rangle_p$

(ii) $\langle f,\alpha g\rangle_p=\alpha^{p-1}\cdot\langle f,g\rangle_p$

(iii)

$$\langle f,\alpha g+\beta h\rangle_p=\sum_{k=0}^{p-1}{}^{p-1}C_k\cdot\alpha^{p-k-1}\cdot\beta^k\cdot\left\langle f,g^{p-k-1}h^k\right\rangle.$$

Proof: (i) is a direct consequence of axioms (2)(i) and (2)(ii) while (ii) follows from axioms (2)(ii) and (2)(iii). (iii) is proved as follows:

$$\langle f,\alpha g+\beta h\rangle_p=\left\langle f,(\alpha g+\beta h)^{p-1}\right\rangle$$

$$=\left\langle (\alpha g+\beta h)^{p-1},f\right\rangle$$

$$=\left\langle \sum_{k=0}^{p-1}{}^{p-1}C_k\left(\alpha g\right)^{p-k-1}\cdot\left(\beta h\right)^k,f\right\rangle$$

(using the binomial expansion)

$$=\sum_{k=0}^{p-1}{}^{p-1}C_k\alpha^{p-k-1}\cdot\beta^k\left\langle g^{p-k-1}h^k,f\right\rangle$$

$$=\sum_{k=0}^{p-1}{}^{p-1}C^k\alpha^{p-k-1}\beta^k\cdot\left\langle f,g^{p-k-1}h^k\right\rangle$$

□

It should be noted that only two terms of the last relation survive when $p=2$. This gives

$$\langle f,\alpha g+\beta h\rangle_2=\alpha\cdot\langle f,g\rangle_2+\beta\langle f,h\rangle_2$$

in consonance with the property of the inner product on the real $L^2$ space.

Defining the $L^p$-norm as $\|f\|_p=\langle f,f\rangle_p^{1/p}$, for $f\in L^p$, we can establish relations for $\|f+g\|_p^p$ and $\|f-g\|_p^p$, their sum and difference which could include and generalize the well known parallelogram and polarization identities for $p=2$. The following identities address these:

If $f,g\in L^p$, then

(i) $\|f+g\|_p^p=\sum_{k=0}^{p-1}{}^{p-1}C_k\cdot\left\langle f+g,f^{p-k-1}g^k\right\rangle$

(ii)

$$\|f-g\|_p^p=\sum_{k=0}^{p-1}{}^{p-1}C_k\cdot(-1)^k\cdot\left\langle f-g,f^{p-k-1}g^k\right\rangle$$

Proof:

Recall that $\|f\|_p^p=\langle f,f\rangle_p$. Thus

$$\|f+g\|_p^p=\langle f+g,f+g\rangle_p$$

$$=\langle f,f+g\rangle_p+\langle g,f+g\rangle_p$$

$$=\sum_{k=0}^{p-1} {}^{p-1}C_k \left\langle f, f^{p-k-1} g^k \right\rangle + \sum_{k=0}^{p-1} {}^{p-1}C_k \left\langle g, f^{p-k-1} g^k \right\rangle$$

by appropriate uses of the result (iii) in section III. Thus,

$$\|f+g\|_p^p = \sum_{k=0}^{p-1} {}^{p-1}C_k \left\langle f+g, f^{p-k-1} g^k \right\rangle .$$

The same procedure obtains for $\|f-g\|_p^p$

□

Remarks: By adding and subtracting the two results above, on obtains two identities that include and generalize the parallelogram and polarization identities (for *p=2*) respectively. The following computations are given for a better understanding of the results of section III.

## IV. COMPUTATIONS

a. P = 2

$$\|f+g\|_2^2 = \sum_{k=0}^{1} {}^{1}C_k \left\langle f+g, f^{1-k} g^k \right\rangle$$

$$=\sum_{k=0}^{1} {}^{1}C_k \left( \left\langle f, f^{1-k} g^k \right\rangle + \left\langle g, f^{1-k} g^k \right\rangle \right)$$

$$=\left( \left\langle f, f \right\rangle + \left\langle g, f \right\rangle \right) + \left( \left\langle f, g \right\rangle + \left\langle g, g \right\rangle \right)$$

$$=\left\langle f, f \right\rangle + \left\langle f, g \right\rangle + \left\langle f, g \right\rangle + \left\langle g, g \right\rangle$$

$$=\|f\|_2^2 + 2\cdot\left\langle f, g \right\rangle + \|g\|_2^2$$

In the same way

$$\|f-g\|_2^2 = \|f\|_2^2 - 2\cdot\left\langle f, g \right\rangle + \|g\|_2^2$$

Adding and subtracting, one has the parallelogram and polarization identities respectively.

b. P = 4

$$\|f+g\|_4^4 = \sum_{k=0}^{3} {}^{3}C_k \left\langle f+g, f^{3-k} g^k \right\rangle$$

$$= \|f\|_4^4 + 4\cdot\left\langle f^3, g \right\rangle + 6\cdot\left\langle f^2, g^2 \right\rangle + 4\left\langle f, g^3 \right\rangle + \|g\|_4^4$$

In the same way

$$\|f-g\|_4^4 = \|f\|_4^4 - 4\cdot\left\langle f^3, g \right\rangle + 6\cdot\left\langle f^2, g^2 \right\rangle - 4\left\langle f, g^3 \right\rangle + \|g\|_4^4$$

These give

$$\|f+g\|_4^4 + \|f-g\|_4^4 = 2\|f\|_4^4 + 12\left\langle f^2, g^2 \right\rangle + 2\|g\|_4^4$$

and

$$\|f+g\|_4^4 - \|f-g\|_4^4 = 8\left( \left\langle f^3, g \right\rangle + \left\langle f, g^3 \right\rangle \right).$$

c. P = 6

Here we obviously have

$$\|f+g\|_6^6 = \|f\|_6^6 + 6\left\langle f^5, g \right\rangle + 15\left\langle f^4, g^2 \right\rangle + 20\left\langle f^3, g^3 \right\rangle + 15\left\langle f^2, g^4 \right\rangle + 6\left\langle f, g^5 \right\rangle + \|g\|_6^6 ,$$

$$\|f-g\|_6^6 = \|f\|_6^6 - 6\langle f^5, g\rangle + 15\langle f^4, g^2\rangle - 20\langle f^3, g^3\rangle + 15\langle f^2, g^4\rangle - 6\langle f, g^5\rangle + \|g\|_6^6,$$

$$\|f+g\|_6^6 + \|f-g\|_6^6 = 2\|f\|_6^6 + 30\left(\langle f^4, g^2\rangle + \langle f^2, g^4\rangle\right) + 2\|g\|_6^6,$$

and

$$\|f+g\|_6^6 - \|f-g\|_6^6 = 12\langle f^5, g\rangle + 40\langle f^3, g^3\rangle + 12\langle f, g^5\rangle.$$

Remarks: Before going to the consideration of the $L^p$-norm induced by this inner product it is to be noted that for all $p \in 2$ I/V, there exists a function, $C_p(f,g)$, of members of $L^p$, such that

$$\|f+g\|_p^p + \|f-g\|_p^p = 2\|f\|_p^p + C_p(f,g) + \|g\|_p^p.$$

The following gives the structure of $C_p(f,g)$ and some of its properties.

## V. PROPERTIES

For every $f, g \in L^p$, the following holds:

(i) $C_p(f,g) = 2 \cdot \sum_{k \in 2I/V}^{p-2} {}^pC_k \langle f^{p-k}, g^k\rangle$

(ii) $C_p(f,g) > 0$ and $C_2(f,g) \equiv 0$

(iii) $C_p(-f,g) = C_p(f,g)$

(iv) $C_p(f,g) = C_p(g,f)$

(v) $C_p(\alpha f, \alpha g) = |\alpha|^p \cdot C_p(f,g), \alpha \in R$

(vi) $$2\sum_{k \in 2I/V}^{p-2} {}^pC_k |\alpha|^k = |\alpha+1|^p + |\alpha-1|^p - 2(|\alpha|^p + 1), \alpha \in R$$

and in particular

$$\sum_{k \in 2I/V}^{p-2} {}^pC_k = 2^{p-1} - 2.$$

Proof:

(i) Note that

$$C_p(f,g) = \|f+g\|_p^p + \|f-g\|_p^p - 2\|f\|_p^p - 2\|g\|_p^p$$

Thus

$$C_p(f,g) = \sum_{k=0}^{p-1} {}^{p-1}C_k \langle f+g, f^{p-k-1}g^k\rangle + \sum_{k=0}^{p-1} {}^{p-1}C_k \cdot (-1)^k \langle f-g, f^{p-k-1}g^k\rangle - 2\|f\|_p^p - 2\|g\|_p^p$$

$$= \sum_{k=0}^{p-1} {}^{p-1}C_k \langle f, f^{p-k-1}g^k\rangle + \sum_{k=0}^{p-1} {}^{p-1}C_k \langle g, f^{p-k-1}g^k\rangle +$$

$$\sum_{k=0}^{p-1} {}^{p-1}C_k \cdot (-1)^k \langle f, f^{p-k-1}g^k\rangle - \sum_{k=0}^{p-1} {}^{p-1}C_k \cdot (-1)^k \langle g, f^{p-k-1}g^k\rangle - 2\|f\|_p^p - 2\|g\|_p^p$$

$$= \sum_{k=1}^{p-1} {}^{p-1}C_k \cdot \left\langle f, f^{p-k-1} g^k \right\rangle + \sum_{k=0}^{p-2} {}^{p-1}C_k \left\langle g, f^{p-k-1} g^k \right\rangle$$
$$+ \sum_{k=1}^{p-1} {}^{p-1}C_k \cdot (-1)^k \left\langle f, f^{p-k-1} g^k \right\rangle - \sum_{k=0}^{p-2} {}^{p-1}C_k \cdot (-1)^k \left\langle g, f^{p-k-1} g^k \right\rangle$$
$$= \sum_{k=0}^{p-1} {}^{p-1}C_k \cdot \left\langle f, f^{p-k-1} g^k \right\rangle + \sum_{k=1}^{p-1} {}^{p-1}C_{k-1} \cdot \left\langle g, f^{p-k} g^{k-1} \right\rangle$$
$$+ \sum_{k=0}^{p-1} {}^{p-1}C_k \cdot (-1)^k \left\langle f, f^{p-k-1} g^k \right\rangle - \sum_{k=1}^{p-1} {}^{p-1}C_{k-1} \cdot (-1)^{k-1} \left\langle g, f^{p-k} g^{k-1} \right\rangle$$

(By index shifts in the second and fourth sums.)

$$= \sum_{k=1}^{p-1} \left( {}^{p-1}C_k + {}^{p-1}C_k \cdot (-1)^k \right) \left\langle f, f^{p-k-1} g^k \right\rangle + \sum_{k=1}^{p-1} \left( {}^{p-1}C_{k-1} - {}^{p-1}C_{k-1} \cdot (-1)^{k-1} \right) \left\langle g, f^{p-k} g^{k-1} \right\rangle$$

$$= \sum_{k=0}^{p-1} \left( {}^{p-1}C_k + {}^{p-1}C_k \cdot (-1)^k \right) \left\langle f^{p-k} g^k \right\rangle + \sum_{k=1}^{p-1} \left( {}^{p-1}C_{k-1} + {}^{p-1}C_{k-1} \cdot (-1)^k \right) \left\langle f^{p-k}, g^k \right\rangle$$

(Since $\left\langle f^{\alpha}, f^{\beta} g^{\gamma} \right\rangle = \left\langle f^{\alpha+\beta}, g^{\gamma} \right\rangle$ and $(-1)^{k-1} = -(-1)^k$.)

$$= \sum_{k=1}^{p-1} \left[ {}^{p-1}C_k + {}^{p-1}C_k (-1)^k + {}^{p-1}C_{k-1} + {}^{p-1}C_{k-1} (-1)^k \right] \left\langle f^{p-k}, g^k \right\rangle$$

$$= \sum_{k=1}^{p-1} \left[ \left( {}^{p-1}C_k + {}^{p-1}C_{k-1} \right) + \left( {}^{p-1}C_k + {}^{p-1}C_{k-1} \right) (-1)^k \right] \left\langle f^{p-k}, g^k \right\rangle$$

$$= \sum_{k=0}^{p-1} \left[ {}^{p}C_k + {}^{p}C_k \cdot (-1)^k \right] \left\langle f^{p-k}, g^k \right\rangle$$

$$= 2 \cdot \sum_{k \in 2IV}^{p-2} {}^{p}C_k \left\langle f^{p-k}, g^k \right\rangle,$$

since ${}^{p}C_k + {}^{p}C_k (-1)^k \equiv 0$ for odd values of $k$ and ${}^{p}C_k + {}^{p}C_k (-1)^k = 2 \cdot {}^{p}C_k$ for $k \in 2IN$. (ii), (iii), (iv), (v) follow from either

$$C_p(f, g) = 2 \cdot \sum_{k \in 2IV}^{p-2} {}^{p}C_k \left\langle f^{p-k}, g^k \right\rangle$$

or

$$C_p(f, g) = \|f+g\|_p^p + \|f-g\|_p^p - 2\|f\|_p^p - 2\|g\|_p^p.$$

We prove (vi) next. (vi) Observe that $C_p\left(f,\alpha f\right)=2\cdot\sum_{k\in 2I/V}^{p-2}{}^{p}C_k\alpha^k\cdot\left\|f\right\|_p^p$ and, from the second definition,

$$C_p\left(f,\alpha f\right)=\left[\left|\alpha+1\right|^p+\left|\alpha-1\right|^p-2\left(\left|\alpha\right|^p+1\right)\right]\left\|f\right\|_p^p.$$

Equating these two expressions for $C_p\left(f,\alpha f\right)$ gives (vi) for $\alpha\in R.$

If $\alpha=1,$ we have $\sum_{k\in 2I/V}^{p-2}{}^{p}C_k=2^{p-1}-2$ . □

## VI. ON THE AXIOMS OF A NORM

It would be appropriate at this point to justify the consideration of only values of *p* in *2IN* and to show the $\left\|f\right\|_p=\left\langle f,f\right\rangle_p^{1/p}$ is indeed a norm in the sense that (i) $\left\|f\right\|_p\geq 0$ and $\left\|f\right\|_p=0$ if, and only if, $f=0$

(ii) $\left\|\alpha f\right\|_p=\left|\alpha\right|\cdot\left\|f\right\|_p$

(iii) $\left\|f+g\right\|_p\leq\left\|f\right\|_p+\left\|g\right\|_p$, for $f,g\in R$ and all $p\in 2IN.$ The justification of $p\in 2IN$ and the proofs of the three axioms for $\left\|f\right\|_p=\left\langle f,f\right\rangle_p^{1/p}$ follow next.

Axioms (i) is a direct consequence of the definition of $\left\|f\right\|_p$ and (2)(iv), while axiom (ii) is established as follows:

$$\left\|\alpha f\right\|_p^p=\left\langle\alpha f,\left(\alpha f\right)\right\rangle_p=\alpha^p\left\langle f,f\right\rangle_p$$

$$=\left|\alpha\right|^p\cdot\left\langle f,f\right\rangle_p;\text{ for } p\in 2IN$$

$$=\left|\alpha\right|^p\cdot\left\|f\right\|_p^p$$

and the result follows. The second to the last step above would had been impossible if p were not strictly taken from $2I/V,$ and this nullifies the possibility of making $L^p$ an inner product space for either $p\in IN$ or $1<p<\infty$ since in this case $\alpha^p\neq\left|\alpha\right|^p$. This serves as the justification for the consideration of only $p\in 2IN$ in the construction of the inner production $L^p$ function spaces.

Another question that could be raised is whether the norm generated from $\left\langle\cdot,\cdot\right\rangle_p$ is indeed the $L^p$-norm of functional analysis. An affirmative answer could be deduced from the following computations:

$$\left\langle f,f\right\rangle_p=\left\langle f,f^{p-1}\right\rangle$$

$$=\int_{[a,b]}f\cdot f^{p-1}d\mu=\int_{[a,b]}f^p d\mu$$

(an integral representation of $\left\langle\cdot,\cdot\right\rangle$ on $L^p$

$$=\int_{[a,b]}\left|f\right|^p d\mu\text{ (since } p\in 2IN\text{ )}$$

$$=\left\|f\right\|_p^p$$

Remarks: The most important of these three axioms are (ii) and (iii). In the proof axiom (ii) the necessary and sufficient values of $p$ for which the new inner product $\left\langle\cdot,\cdot\right\rangle_p$ induces a consistent $L^p$-norm is derived and found to be all $p\in 2IN$, in which we saw that the first member of this countable family of inner product $L^p$ spaces is actually the well behaved $L^2$ space.

An investigation on how to establish axiom (iii),*i.e.*,

$$\left\|f+g\right\|_{p\in 2IN}\leq\left\|f\right\|_{p\in 2IN}+\left\|g\right\|_{p\in 2IN},$$

with the use of $\left\langle\cdot,\cdot\right\rangle_p$ takes us to a consideration of the expression derived for $\left\|f+g\right\|_{p\in 2IN}^p$ in section **III** (supra). Since this expression contains terms in terms of inner product, it would be appropriate to look for their common bound in the fashion of the Cauchy-Bunyakovskii-Schwarz Inequality, $\left|\langle f,g\rangle\right| \le \|f\|_2 \|g\|_2$, that gives a bound for $\langle f,g\rangle$ used in the proof of $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$. The inequality expressing this common bound in the expression in $\|f+g\|_{p\in 2IN}$ would be expected to bore down to $\left|\langle f,g\rangle\right| \le \|f\|_2 \|g\|_2$ at $p=2$. This auxiliary inequality and it remarkable proof is detailed below.

## VII. INEQUALITY

$$\left|\left\langle f^{p-k}, g^k\right\rangle\right| \le \|f\|_p^{p-k} \cdot \|g\|_p^k$$

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holds for all $f,g\in L^p, k\in IN, p\in 2IN.$

Proof:

Observe that by the Cauchy-Bunyakovskii-Schwarz Inequality we have

$\left|\langle f,g\rangle\right| \le \|f\|_2 \|g\|_2$. It then follows that

$$\left|\left\langle f^{p-k}, g\right\rangle\right| \le \left\|f^{p-k}\right\|_2 \cdot \left\|g^k\right\|_2$$

(I) The only thing that need to be shown is the fact that

$$\left\|f^{p-k}\right\|_2 \cdot \left\|g^k\right\|_2 = \|f\|_p^{p-k} \cdot \|g\|_p^k,$$

and is as follows:

$$\left\|f^{p-k}\right\|_2 \cdot \left\|g^k\right\|_2 = \left(\int_{[a,b]} |f|^{2(p-k)}\, d\mu\right)^{\frac{1}{2}} \cdot \left(\int_{[a,b]} |g|^{2k}\, d\mu\right)^{\frac{1}{2}}$$

$$= \left[\left(\int_{[a,b]} |f|^{2(p-k)}\, d\mu\right)^{\frac{1}{2(p-k)}}\right]^{p-k} \cdot \left[\left(\int_{[a,b]} |g|^{2k}\, d\mu\right)^{\frac{1}{2k}}\right]^{k}$$

$$= \|f\|_{2(p-k)}^{p-k} \cdot \|g\|_{2k}^{k}.$$

Since $k\in IN$, we can set $p=2k\in 2IN$, thus making the above subscripts, $2(p-k)$ and $2k$, to coincide and be equal to *p*. Hence

$$\left\|f^{p-k}\right\|_2 \cdot \left\|g^k\right\|_2 = \|f\|_{2(p-k)}^{p-k} \cdot \|g\|_{2k}^{k}$$

$$= \|f\|_p^{p-k} \cdot \|g\|_p^k.$$

Whence

$$\left|\left\langle f^{p-k}, g^k\right\rangle\right| \le \|f\|_p^{p-k} \cdot \|g\|_p^k$$

∎

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Remarks: Setting $p=2, k=1$ gives the Cauchy-Bunyakovskii-Schwarz Inequality as expected.

However, the auxiliary inequality above is not only of interest as an extension of the Cauchy-Bunyakovskii-Schwarz Inequality, but also a very important tool in later proofs, most especially in the inner product proof of $\|f+g\|_{p\in 2IN} + \|g\|_{p\in 2IN}$ (as given below) and in the properties of $\langle\cdot,\cdot\rangle_p$:

$$\|f+g\|_{p\in 2IN} \le \|f\|_{p\in 2IN} + \|g\|_{p\in 2IN}$$

for all the inner product $L^p$ function spaces.

Proof:

From section III, we have

$$\|f+g\|_p^p = \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p-1}C_k \left\langle f^{p-k}, g^k \right\rangle + \sum_{k=0}^{p-2} {}^{p-1}C_k \left\langle f^{p-k-1}, g^{k+1} \right\rangle + \|g\|_p^p$$

$$= \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p-1}C_k \left\langle f^{p-k}, g^k \right\rangle + \sum_{k=1}^{p-1} {}^{p-1}C_{k-1} \left\langle f^{p-k}, g^k \right\rangle + \|g\|_p^p$$

(by an index shift in the second sum)

$$= \|f\|_p^p + \sum_{k=1}^{p-1}\left( {}^{p-1}C_k + {}^{p-1}C_{k-1}\right)\left\langle f^{p-k}, g^k \right\rangle + \left\|g_p^p\right\|$$

$$= \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p}C_k \left\langle f^{p-k}, g^k \right\rangle + \|g\|_p^p$$

$$\leq \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p}C_k \left|\left\langle f^{p-k}, g^k \right\rangle\right| + \|g\|_p^p$$

$$\leq \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p}C_k \left\|f\right\|_p^{p-k} \cdot \|g\|_p^p + \|g\|_p^p$$

$$= \sum_{k=0}^{p} {}^{p}C_k \left\|f\right\|_p^{p-k} \cdot \|g\|_p^k = \left(\|f\|_p + \|g\|_p\right)^p$$

□

Completeness of $L^{p\in 2IN}$ as inner product spaces follows form it norm-metric. The author now goes straight to the business of generating orthogonal polynomials in $L^p\left(\left[-1,1\right]\right)$, say.

## VIII. ORTHOGONALITIES

The existence of an inner product on a particular function or sequence space naturally leads to a consideration of the concept of orthogonality of members of the space or that of it subspaces. The central concept of orthogonality, which is peculiar to only inner product spaces, is a generalization of the condition of the condition of perpendicularity of vectors in elementary vector algebra. To start with let us consider the classical case: we say two non-zero members, *x, y* of an inner product admissible space are orthogonal iff $\left\langle x, y\right\rangle = 0$ [2]. One may want to extend this concept to $\left\langle \cdot,\cdot\right\rangle_p$. This extension thus result to two kinds of orthogonalities.

Definitions:

(i) Let $f, g \in L^p$. We say *f* is orthogonal with respect to *g* if $\left\langle f, g\right\rangle_p = 0$. Since this does not necessarily imply $\left\langle g, f\right\rangle_p = 0$, we have a second kind of orthogonality in $L^p$:

(ii) If both $\left\langle f, g\right\rangle_p = 0$ and $\left\langle g, f\right\rangle_p = 0$ hold, we say *f* and *g* are completely orthogonal. The following example might be of importance.

Example:

Let $e_0(t) = 2^{-1/p}$ and $e_1(t) = \left(\dfrac{p+1}{2}\right)^{1/p} t$.

Then $\left\langle e_0(t), e_1(t)\right\rangle_p \neq 0$ but $\left\langle e_1(t), e_0(t)\right\rangle_p = 0$ in $L^p\left([-1,1\right), p \in 2IV$.

Observe that the two kinds of orthogonalities coincide at *p=2*, for obvious reasons. Before entering into the subject of Gram-Schmidt Orthonormalisation Procedure, let us establish some useful properties of orthogonality and linear independence in $\left\langle \cdot,\cdot\right\rangle_p$.

a. Properties

(a) Let $f_k \neq 0$ and $\left\langle f_{k,} f_i\right\rangle_p = 0,\ k \neq i$ then $\{f_k\}_{k=1}^n$ is a linearly independent set.

(b) $\left\langle f, g\right\rangle_p = \left\langle f, h\right\rangle_p$, then $g = h; f, g, h \in L^p$.

(c) Given that $\langle f_n, g\rangle_p = 0$ and $f_n \to f$, then $\langle f, g\rangle_p = 0.$

(d) Continuity of $\langle \cdot,\cdot\rangle_p$: If $f_n \to f$ and $g_n \to g$, then $\langle f_n, g_n\rangle_p \to \langle f, g\rangle_p$.

(e) An orthonormal set is a linearly independent set.

Proofs:

(a) Consider $\sum_{k=1}^{n} \alpha_k f_k = 0,$

$$\left\langle \sum_{k=1}^{n} \alpha_k f_{k,} f_i \right\rangle_p = \langle 0, f_i\rangle_p$$

$$= \langle 0.0, f_i\rangle_p$$

$$= 0 \cdot \langle 0 \cdot f_i\rangle_p = 0.$$

$$\therefore \sum_{k=1}^{n} \alpha_k \langle f_k, f_i\rangle_p = 0$$

$$\Rightarrow \alpha_k \langle f_k, f_k\rangle_p = 0$$

$$\alpha_k \cdot \|f_k\|_p^p = 0$$

$$\Rightarrow \alpha_k = 0, \text{ as } f_k \neq 0.$$

---

(b) If $\langle f, g\rangle_p = \langle f, h\rangle_p$, then

$$\langle f, g^{p-1}\rangle = \langle f, h^{p-1}\rangle$$

$$\Rightarrow g^{p-1} = h^{p-1}; \text{ i.e. } g = h.$$

(c) Consider

$$\left|\langle f_n, g\rangle_p - \langle f, g\rangle_p\right| = \left|\langle f_n - f, g\rangle_p\right|$$

$$= \left|\langle f_n - f, g^{p-1}\rangle\right|,$$

since $\|f_n - f\|_p \to 0,$ as $n \to \infty$.

$\therefore \quad \left|\langle f_n, g\rangle_p - \langle f, g\rangle\right| = 0$

*i.e.*, $\quad \left|\langle f, g\rangle\right| = 0$

$\quad \Rightarrow \langle f, g\rangle = 0$

(d) Suppose that

$$\|f_n - f\| \to 0, \|g_n - g\|_p \to 0, \text{ as } n \to \infty.$$

The latter convergence also implies

$$\left\|g_n^{p-1} - g^{p-1}\right\|_p \to 0, n \to \infty \text{ for } p \in 2IV.$$

Now,

$$\left|\langle f_n, g_n\rangle_p - \langle f, g\rangle_p\right| = \left|\langle f_n, g_n\rangle_p - \langle f_n, g\rangle_p + \langle f_{n,} g\rangle_p - \langle f, g\rangle_p\right|$$

$$= \left|\langle f_n, g_n^{p-1}\rangle - \langle f_n, g^{p-1}\rangle + \langle f_n, g^{p-1}\rangle - \langle f, g^{p-1}\rangle\right|$$

$$= \left|\langle f_n, g_n^{p-1} - g^{p-1}\rangle + \langle f_n - f, g^{p-1}\rangle\right|$$

$$\leq \left|\langle f_n, g_n^{p-1} - g^{p-1}\rangle\right| + \left|\langle f_n - f, g^{p-1}\rangle\right|$$

$$\leq \|f_n\|_p \cdot \left\|g_n^{p-1} - g^{\,p-1}\right\|_p + \|f_n - f\|_p \cdot \|g\|_p^{p-1} \to 0$$

(e) Let $\{e_k\}_{k=1}^{n}$ be an orthonormal set and consider $\sum_{k=1}^{n}\alpha_k e_k = 0,$ then

$$\left\langle \sum_{k=1}^{n}\alpha_k e_k e_j \right\rangle = 0, \quad 1 \le j \le n$$

$$\therefore \quad \sum_{k=1}^{n}\alpha_k \left\langle e_k, e_j \right\rangle_p = 0$$

$$\alpha_k \left\langle e_k, e_k \right\rangle_p = 0$$
$$\alpha_k \left\| e_k \right\|_p^p = 0$$
$$\alpha_k (1) = 0$$
$$\alpha_k = 0$$

□

Let $\{e_m\}_{m=1,2,....,n}$ be an orthonormal sequence in $L^p$ and

$f \in span\{e_1, e_2, ...., e_n\}$, then

$f = \sum_{m=1}^{n}\alpha_m e_m$, for some $\alpha_m \in R$.

Thus,

$$\left\langle f, e_k \right\rangle_p = \left\langle \sum_{m=1}^{n}\alpha_m e_m, e_k \right\rangle_p$$
$$= \sum_{m=1}^{n}\alpha_m \left\langle e_m, e_k \right\rangle_p$$
$$= \alpha_k \cdot \left\langle e_k, e_k \right\rangle_p$$
$$= \alpha_k \cdot \left\| e_k \right\|_p^p = \alpha_k \cdot (1) = \alpha_k$$

Hence

$f = \sum_{m=1}^{n}\left\langle f, e_m \right\rangle_p \cdot e_m$ and $\alpha_k = \left\langle f, e_k \right\rangle_p$

is its Fourier coefficient.

Let us now generate the orthonormal sequence $\{e_k\}_{k\in\{0\}\cup IV}$ from a given linearly independent set $\{f_i\}_{i=0,1,2,....}$. The procedure is given as follows: The first element, $e_0$, is chose as $e_o = \frac{f_0}{\|f_o\|_p}, e_1 = \frac{V_1}{\|V_1\|_p}$ in which $V_1 = f_1 - \left\langle f_1, e_k \right\rangle_p \cdot e_k$, while the rest are $e_n = \frac{V_n}{\|V_n\|_p}$ where

$$V_n = f_n - \sum_{k=1}^{n-1}\left\langle f_n, e_k \right\rangle_p \cdot e_k, \; n = 2, 3, ...$$

Let us show that $V_n$ is indeed orthogonal to any of $\{e_k\}_{k=1}^{n-1}$, as follows:

$$\left\langle V_n e_m \right\rangle_p = \left\langle f_n - \sum_{k=1}^{n-1}\left\langle f_n e_k \right\rangle_p \cdot e_k, e_m \right\rangle_p$$

$$= \left\langle f_n, e_m \right\rangle_p - \sum_{k=1}^{n-1}\left\langle f_n, e_k \right\rangle_p \cdot \left\langle e_k, e_m \right\rangle_p$$

$$= \left\langle f_n, e_m \right\rangle_p - \left\langle f_n, e_m \right\rangle_p \cdot \left\langle e_m, e_m \right\rangle_p$$

$$= \left\langle f_n e_m \right\rangle_p - \left\langle f_n, e_m \right\rangle_p \cdot \left\| e_m \right\|_p^p$$

$$= \left\langle f_n, e_m \right\rangle_p - \left\langle f_n, e_m \right\rangle_p = 0$$

Let us now exhibit a set of orthonormal polynomials in $L^p([-1,1])$ for all $p \in 2IV$, by the Gram-Schmidt procedure [8] above.

b. Construction Of Higher Orthonormal Polynomials In $L^p([-1,1])$.

We first consider a linearly independent set $\{1, t, t^2, ...., t^k, ....,\}$ and derive the polynomials from this set, which can be re-written as $\{f_k(t) = t^k; k \in \{0\} \cup IV\}$. For the first of these polynomials, set $V_0(t) = f_0(t) = 1,$ then

$$\left\| V_o(t) \right\|_p = \left[ \int_{[-1,1]} \left( V_0(t) \right)^p d\mu \right]^{1/p} = 2^{1/p}$$

Thus from $e_0(t)=\dfrac{V_0}{\|V_0\|_p}$ we have the first of these polynomials as

$$e_0(t)=\left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Note that for $p=2, e_0(t)=\sqrt{\dfrac{1}{2}}.$

For $e_1(t)$, we find

$$\begin{aligned} V_1(t) &= f_1-\langle f_1,e_k\rangle_p\cdot e_k \\ &= t-0\cdot e_k;\ \text{as } \langle f_1,e_k\rangle_p=0 \\ &= t \end{aligned}$$

This implies that

$$\|V_1\|_p=\left[\int_{[-1,1]}\left(V_1(t)\right)^p\,d\mu\right]^{\frac{1}{p}}=\left(\frac{2}{p+1}\right)^{\frac{1}{p}}$$

and since

$$e_1(t)=\frac{V_1(t)}{\|V_1\|_p},$$

we then have that

$$e_1(t)=\left(\frac{p+1}{2}\right)^{\frac{1}{p}}t$$

It is observed that for $p=2$,

$$e_1(t)=\sqrt{\frac{3}{2}}t\ .$$

For $\underline{n=2}$; we use $V_2(t)=f_2-\sum_{k=0}^{1}\langle f_2,e_k\rangle_p\cdot e_k$ to have that

$$V_2(t)=t^2-\frac{1}{3}$$

since

$$\|V_2(t)\|_p=\frac{1}{3}\left(2\sum_{k=0}^{p}(-1)^k\,\frac{3^{p-k}\cdot{}^pC_k}{2p-2k+1}\right)^{\frac{1}{p}},$$

then

$$e_2(t)=\left(\frac{1}{2\sum_{k=0}^{p}(-1)^k\cdot\dfrac{3^{p-k}\cdot{}^pC_k}{2p-2k+1}}\right)^{\frac{1}{p}}\cdot\left(3t^2-1\right)$$

is the third orthonormal polynomial in the inner product $L^p([-1,1])$ function space, which reduces to $e_2(t)=\sqrt{\dfrac{5}{8}}\left(3t^2-1\right)$, in the classical case of $p=2$.

For $\underline{n=3}$: We have $V_3(t)=f_3-\sum_{k=0}^{2}\langle f_3,e_k\rangle_p\cdot e_k$. Thus $V_3(t)=\dfrac{1}{p+3}\left[(p+3)t^3-(p+1)t\right]$ and

$$\|V_3\|_p=\frac{1}{p+3}\left[2\sum_{k=0}^{p}(-1)^k\cdot\frac{(p+3)^{p-k}\cdot(p+1)^k\cdot{}^pC_k}{3p-2k+1}\right]^{\frac{1}{p}}$$

The next orthonormal polynomial in $L^p([-1,1])$ is thus

$$e_3(t)=\left(\frac{1}{2\sum_{k=0}^{p}(-1)^k\cdot\frac{(p+3)^{p-k}\cdot(p-1)^k\cdot{}^pC_k}{3p-2k+1}}\right)^{\frac{1}{p}}\left[(p+3)t^3-(p-1)t\right]$$

This gargantuan expression reduces to

$$e_3(t)=\sqrt{\frac{7}{8}\left(5t^3-3t\right)}$$

when $p=2$.

Anyone not devoid of patience can exhibit as many of these orthonormal polynomials as possible. At every point of these constructions we can derive the orthonormal sequence $\{e_k(t)\}_{k=0,1,...}$ in term of the classical Legendre Polynomials when we set $p=2$. But more than Legendre Polynomials are arrived at with $\langle\cdot,\cdot\rangle_p$ in $L^p([-1,1])$ for all $p\in 2IV$. For example for $p=4$:

$$e_0(t)=\left(\frac{1}{2}\right)^{\frac{1}{4}},$$

$$e_1(t)=\left(\frac{5}{2}\right)^{\frac{1}{4}}t$$

$$e_2(t)=\left(\frac{35}{96}\right)^{\frac{1}{4}}\left(3t^2-1\right)$$

$$e_3(t)=\left(\frac{1287}{7008}\right)^{\frac{1}{4}}\left(7t^3-5\right),...$$

From the structure of each $e_n(t)$ above it is obvious that it can be expressed as

$$e_n(t)=\alpha_{(n,p)}\cdot P_{(n,p)}(t)$$

for each $n\in\{0\}\cup IV$ and $p\in 2IV$, where $\alpha_{(n,p)}\in R$ for each *n* and *p* and $P_{(n,p)}(t)$ an *n*-degree polynomial in $t\in[-1,1]$, with the properties that

$$\alpha_{(n,2)}=\sqrt{\frac{2n+1}{2}}$$

and $P_{(n,2)}(t)=$ Legendre Polynomials.

The generalization afforded by $\langle\cdot,\cdot\rangle_p$ could also be sought for other orthogonal polynomials in $L^2([0,\infty)),L^2((-\infty,\infty))$ and their representation, properties, zeros, generating functions, derivatives about the origin, recurrence relations,…studied in the style of Szegö's Orthogonal Polynomials [9].

## IX. CONCLUSION

The theory of inner product space and it applications in the $L^p$-spaces is worth a second look since the inner product $\langle\cdot,\cdot\rangle$, on $L^2$ is one in a countable.

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