Einstein's 1905 Paper on $E=mc^2$

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ABSTRACT
It is well-known that Einstein's first paper on $E = mc^2$ as published in the *Annalen der Physik* in 1905 is problematic in that it suffers from the error of circular reasoning. This means that it uses as one of its premises a statement which is equivalent to the conclusion of the paper, namely, that $E = mc^2$. This difficulty with the paper has been pointed out by many writers including Max Planck, Herbert Ives, Max Jammer and also biographers of Einstein including Gerald Holton and Arthur I. Miller. Unfortunately, the derivation is repeated today as being correct without any mention of the above criticisms of it. In view of this it seems to us worthwhile to have a clear and as simple as possible explanation of the logical difficulties associated with Einstein's 1905 derivation, and it is to this end that this paper is written. Herewith we present a very simple treatment of the problem which makes absolutely clear the logical difficulties in Einstein's first published work on $E = mc^2$.

KEYWORDS
History of Science; Origins of Special Relativity

INTRODUCTION

Does Einstein's first paper[1] on $E = mc^2$ suffer from the fallacy of circular reasoning? A circular argument, which in philosophical circles is frequently referred to as *petitio principii* or *circulus in probando*, is one that assumes what it is trying to prove. It inserts the conclusion into the premises, as in the simple example, "Jason is the best candidate for the position because Jason is better than all the other candidates. Therefore, Jason is the best candidate for the position." No one will argue that such argumentation is circular. So also, according to many notable scientists, such as Max Planck and the well-known historian of physics, Max Jammer, is Einstein's 1905 paper on $E = mc^2$.[*] Unfortunately, this paper entitled "Does the Inertia of a body depend upon its energy content?" is referenced in scientific articles and textbooks on special relativity as gospel truth coming from the great scientist himself without any reference to the problems existing within it. Others try to defend the paper against the logical criticisms of the above mentioned scientists, however, none successfully, for to do such would necessarily have to refute the fallacy of circular reasoning from which we show quite clearly that Einstein's paper suffers. Most simply, as we show below, the error of circular reasoning in Einstein's paper can be understood from our **Equation 7'**, which relates the total energy of a particle in a moving frame to the total energy of a particle in its rest frame to order $\left(\frac{v}{c}\right)^2$. Einstein's argument essentially rests upon using a more

[*]The list of authoritative figures associated with objections to Einstein's 1905 paper started with Max Planck, the father of the quantum theory. His criticism of Einstein's 1905 work is contained in an important 1907 paper by Planck, which some consider to contain the first generally valid and correct derivation of $E = mc^2$ for matter.[2] Apparently Einstein himself was not very happy with his 1905 paper, and shortly afterwards he published another derivation of mass-energy equivalence.[3] That paper, according to Einstein, is basically a rewriting of a treatment of mass-energy equivalence for light given a few years earlier by Poincaré.[4] In 1952 the American physicist, Herbert Ives, wrote a particularly lucid account of the matter on which most subsequent authors base their analyses.[5] In particular, he spells out quite clearly the circular reasoning used in Einstein's 1905 paper. The noted philosopher and historian of physics, Max Jammer, reiterates Ives arguments in his book, *The Concept of Mass in Classical and Modern Physics*.[2] Other scientists and historians of science, such as Gerald Holton,[6] H. Arzeliés[7] and Arthur I. Miller[8] all reiterate these objections of Planck and Ives.

complicated form of **Equation 7'** to derive $E = mc^2$. However, as our analysis clearly shows, it turns out that $E = mc^2$ is already implicit in **Equation 7'**. Thus the fallacy of circular reasoning.†

We were led to this problem by uncovering purported derivations of $E = mc^2$ that one finds on the notoriously dangerous internet highway of false and misleading information. One can find on there all kinds of discussions of $E = mc^2$ claiming to be scientific, but which essentially repeat Einstein's circular argumentation in one way or another, without any mention of the logical difficulties associated with it, or without any reference to the well-established claims that Einstein's paper suffers from such logical error. One of the worst is the "derivation" of $E = mc^2$ posted by Henry Reich.[11] For many a keen student this situation must be a frustrating state of affairs: trying to logically understand the illogical. For indeed there is something wrong with Einstein's 1905 derivation! To this end, it seems worthwhile to have a concise and clear as possible explanation of Einstein's error, which is what we present here. Our argumentation is essentially the same as Ives' (and repeated by Jammer and others) except that we simplify things by considering a particular special and simple case of that considered in Einstein's paper, namely where the particle ceases to exist after it emits light isotropically in all directions, which, for the special situation considered by us, means emission of a light pulse in one direction along with another light pulse emitted in the opposite direction. This amounts to particle annihilation, something utterly unthinkable at the time in which Einstein wrote his paper, and it enables us to see very clearly the logical difficulties associated with Einstein's first paper on $E = mc^2$.

EINSTEIN'S 1905 ANALYSIS OF $E = mc^2$ APPLIED TO PION ANNIHILATION: A SIMPLIFIED TREATMENT SHOWING THE DIFFICULTIES WITH HIS ANALYSIS

Einstein in **Reference 1** considers a body which at some time emits a light pulse in one direction and simultaneously an identical light pulse in the opposite direction. After the emission of the two light pulses the energy of the body is, by conservation of energy, diminished by an amount equal to the sum of the energies of the two light pulses. A most simple case of this situation is when the body ceases to exist after the emission of the light pulses. An elementary particle process which physically describes this case is neutral pion annihilation into two gamma rays (light pulses), described symbolically by the following reaction:

$$\pi_0 \rightarrow \gamma + \gamma.$$

Here γ represents one of the gamma rays and π_0 represents the neutral pion.

Consider the rest frame S_0 of the pion before it decays into two photons. Call its energy E_0 and let its mass be denoted by m. Its energy after the decay may be taken as zero, since it has ceased to exist. The combined energy of both of the emitted photons is E_γ. The energies and kinetic energies of both of the photons and of the pion before and after the decay are shown in **Table 1**. ($\gamma + \gamma$ in the tables represents both photons going off in opposite directions.)

Now consider the situation from the point of view of an observer moving with speed v relative to the pion. The energies of the pion and photons are denoted with primes to distinguish them from the corresponding quantities in the rest frame S_0. The values of these quantities are shown in **Table 2**. In Einstein's paper the kinetic energy K_0' is denoted by K and the energy of radiation (which in our case is the energy of one of the photons) is denoted by L. Einstein uses the subscripts 0 and 1 to distinguish quantities in the S_0 and S_0' frames, respectively. For the energy of the particle before the emission of radiation he uses the symbol E and he uses H to denote the energy of the particle after the emission of light.

S_0	Energy	Kinetic Energy
π_0 (before)	E_0	0
π_0 (after)	0	0
$\gamma + \gamma$	E_γ	0

Table 1. Values of Relevant Quantities in Rest Frame of Pion.

†It should be pointed out that Einstein gave other (mathematically correct) proofs of the equivalence of mass and energy in the following articles: *Annalen der Physik* **20** 627-633 (1906); *Annalen der Physik* **23** 371-384 (1907); *Bull. Amer. Math. Soc.* **41** 223-230 (1935). For an up-to-date discussion about attempts to understand the meaning of $E = mc^2$ we refer the reader to the recent article by Hecht[9] which contains a useful and relatively complete bibliography on the subject including somewhat tenuous attempts to justify on physical grounds Einstein's circular reasoning argument in his paper.[10]

S_0'	Energy	Kinetic Energy
π_0 (before)	E_0'	K_0'
π_0 (after)	0	0
$\gamma + \gamma$	$E_\gamma' = \frac{E_\gamma}{\sqrt{1-\frac{v^2}{c^2}}}$	0

Table 2. Values of Relevant Quantities in the Frame S_0'.

Clearly the energy of each of the photons in the S_0 frame is $\frac{1}{2}E_\gamma$. Hence the total energy of both photons in S_0 frame is E_γ as listed in **Table 1**. According to Einstein's first equation in his paper, we have that

$$\frac{1}{2}E'_\gamma(\phi) = \frac{1}{2}E_\gamma \frac{1-\frac{v}{c}cos\,(\phi)}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \textbf{Equation 1.}$$

represents the energy of a photon going off at an angle ϕ with respect to the positive x axis when the particle is located at the center of the coordinate system (see **Figure 1**). Thus the combined energy of the two photons going off in opposite directions from one another along the x axis is, in the S_0' frame, given by:

$$E'_\gamma = \frac{1}{2}E'_\gamma(0) + \frac{1}{2}E'_\gamma(\pi) = \frac{1}{2}E_\gamma \frac{1-\frac{v}{c}cos\,(0)}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{1}{2}E_\gamma \frac{1-\frac{v}{c}cos\,(\pi)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E_\gamma}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \textbf{Equation 2.}$$

Now we come to Einstein's assumptions which, in his notation, are

$$H_0 - E_0 = K_0 + C \qquad \textbf{Equation 3.}$$

$$H_1 - E_1 = K_1 + C \qquad \textbf{Equation 4.}$$

Using these assumptions, Einstein is led to conclude that $E_0 = mc^2$. (Precisely, Einstein's conclusion states: *If a body gives off energy* L *in the form of radiation, its mass diminishes by* L/c^2. Clearly, this statement amounts to the assertion that $E_0 = mc^2$, and even more so in our case, ours being a special case of Einstein's treatment where the particle ceases to exist after the emission of radiant energy.)

For our case the second assumption (**Equation 4**) reads

$$0 - 0 = 0 + C \qquad \textbf{Equation 4'.}$$

which implies

$$C = 0 \qquad \textbf{Equation 5.}$$

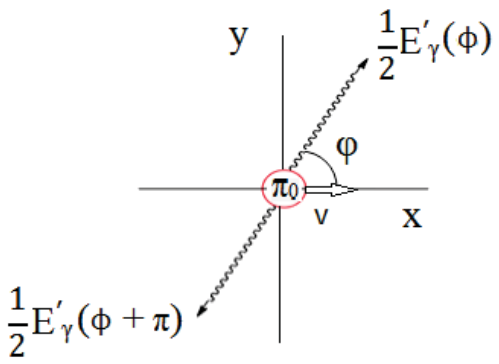

Figure 1. Pion decay, in the S_0' frame, showing two photons going off in opposite directions, the directions making angles ϕ and $\phi + \pi$ with respect to the positive x axis. (For the decay in the S_0 frame, i.e. the rest frame of the pion, the figure looks the same except that $v = 0$ and there are no primes on the $E_\gamma's$.)

The first of Einstein's assumptions (**Equation 3**) in our notation then gives

$$E_0' - E_0 = K_0' + C = K_0' \qquad \textbf{Equation 6.}$$

since we have shown that $C = 0$ for our simplified case. Thus

$$E_0' = K_0' + E_0 = K_0' + E_\gamma \qquad \textbf{Equation 7.}$$

since $E_0 = E_\gamma$ by conservation of energy in the S_0 frame. Hence, for our simple treatment, the constant C is zero and Einstein's assumptions, **Equation 3** and **Equation 4**, amount to just one equation, namely **Equation 7**.

Now conservation of energy in the S_0' frame gives

$$E_0' = \frac{E_\gamma}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad \textbf{Equation 8.}$$

Thus we obtain by using **Equation 7**

$$K_0' = E_\gamma \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right). \qquad \textbf{Equation 9.}$$

This is, for our special case and in our notation, Einstein's second to last equation of his paper. By expanding the term involving the radical as a power series in $\frac{v}{c}$ to order $\left(\frac{v}{c}\right)^2$, we obtain as an approximation:

$$K_0' = \frac{1}{2}\frac{E_\gamma}{c^2} v^2. \qquad \textbf{Equation 10.}$$

This is, for our case and in our notation, Einstein's last equation in his paper.

Let us see how Einstein was led to $E_0 = mc^2$ out of **Equation 7** and **Equation 9**. For $v \ll c$, K_0' must reduce to its nonrelativistic limit, so we have:

$$E_0' = E_\gamma + K_0' = E_\gamma + \frac{1}{2}mv^2 \text{ to order } \left(\frac{v}{c}\right)^2. \qquad \textbf{Equation 7'.}$$

Now expand **Equation 8** in powers of $\left(\frac{v}{c}\right)^2$ to get:

$$E_0' = E_\gamma + \frac{1}{2}\left(\frac{E_\gamma}{c^2}\right)v^2 \text{ to order } \left(\frac{v}{c}\right)^2 \quad . \qquad \textbf{Equation 11.}$$

Equating **Equation 7'** and **Equation 11** we obtain

$$m = \frac{E_\gamma}{c^2}. \qquad \textbf{Equation 12.}$$

This is the argumentation used by Einstein in his paper and it enables him to conclude at the end of his paper what we stated above in italics, namely: *If a body gives off the energy* L *in the form of radiation, its mass diminishes by* L/c^2. In summary, by using **Equation 7'** plus energy conservation, as did Einstein, we have been led to energy mass equivalence i.e. to $E_0 = mc^2$.

Finally we show the logical difficulties associated with Einstein's paper. Specifically we show that not only does **Equation 7'** together with energy conservation imply $E_0 = mc^2$ but also $E_0 = mc^2$ together with energy conservation implies **Equation 7'**. This means (assuming energy conservation) that the statement embodied by **Equation 7'**, i.e. the statement embodied by **Equation 7** to order $\left(\frac{v}{c}\right)^2$, is equivalent to the statement of $E = mc^2$ and hence it cannot be used to derive $E = mc^2$. Let us now show that this is so. From **Equation 11** which comes from **Equation 8** which we were led to by conservation of energy, we obtain:

$$E_0' = E_\gamma + \frac{1}{2}\left(\frac{E_\gamma}{c^2}\right)v^2 + \ ....$$

On the other hand, we also have **Equation 7'** which is

$$E_0' = E_\gamma + \frac{1}{2}mv^2 + \ ....$$

Equating coefficients of the corresponding terms of degree v^2 and using the fact that $E_\gamma = E_0$ gives

$$E = mc^2$$

This shows that **Equation 7** to order $\left(\frac{v}{c}\right)^2$ implies $E_0 = mc^2$. If this is all that we could show, then there would be no logical difficulties in Einstein's paper. But, in fact, as we have already stated above, more can be shown, namely that $E_0 = mc^2$ implies **Equation 7** to order $\left(\frac{v}{c}\right)^2$, i.e. **Equation 7'**. Thus **Equation 7** to order $\left(\frac{v}{c}\right)^2$) is logically equivalent to the statement of $E = mc^2$ and hence it cannot be used to derive $E = mc^2$. This is what the young Einstein unfortunately neglected to realize in his paper.

We now show that $E_0 = mc^2$ together with energy conservation implies **Equation 7'**. As before, we start with **Equation 11** which comes from **Equation 8** which we were led to by conservation of energy. We have:

$$E_0' = E_\gamma + \frac{1}{2}\left(\frac{E_\gamma}{c^2}\right)v^2 + \ ....$$

Combining $E_\gamma = E_0$ (which comes from conservation of energy in the S frame) and $E_0 = mc^2$ we obtain $E_\gamma = mc^2$. Using this in the second term of the above equation gives:

$$E_0' = E_\gamma + \frac{1}{2}mv^2 + \ ....$$

which is **Equation 7'**. Thus **Equation 7'** is equivalent to $E_0 = mc^2$.

CONCLUSIONS

From what we have shown in this paper, it seems clear that there can be no (mathematically correct) argumentation to obtain $E = mc^2$ from considerations starting with a body emitting light isotropically in all directions as is done in **Reference 11.** That this is so, was essentially what Poincaré already pointed out in 1900 when he stated that in order to obtain the result ($E = mc^2$) it is necessary to use a parabolic mirror in order to focus and send light in one direction.[12] It is thus advisable for teachers and

expositors on the subject to steer clear of Einstein's 1905 paper on $E = mc^2$, and instead to follow his 1906 paper, namely **Reference 3**, to explain the famous formula. This was the path followed by Max Born in his well-known book "Einstein's Theory of Relativity".[13] This book which is even suitable for high school students contains a lucid and quite elementary derivation of $E = mc^2$ that is based on **Reference 3**. Born probably was well aware of the logical difficulties with the 1905 paper and thus avoided using it.

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PRESS SUMMARY

It is a well-known and indisputable fact that Einstein's first paper on $E = mc^2$, published in 1905, is problematic in that it suffers from the error of circular reasoning. Despite this defect it is blindly referenced in many scientific articles as the official derivation of the famous formula. Furthermore, the arguments in that paper are still used today by some authors to purportedly derive $E = mc^2$. In view of this disturbing state of affairs, it seems worthwhile to have a clear and as elementary as possible explanation of the erroneous reasoning in Einstein's 1905 paper on $E = mc^2$.