On Closure Properties of Irrational and Transcendental Numbers under Addition and Multiplication

Shekh Mohammed Zahid[a] and Dr. Prasanta Kumar Ray[b]*

[a]*Institute of Mathematics and Applications, Bhubaneswar, Odisha, India*
[b]*Veer Surendra Sai University of Technology, Burla, Odisha, India*

**Student: shekhmohammedzahid@gmail.com*
Mentor: prasanta@iiit-bh.ac.in

ABSTRACT
In the article 'There are Truth and Beauty in Undergraduate Mathematics Research', the author posted a problem regarding the closure properties of irrational and transcendental numbers under addition and multiplication. In this study, we investigate the problem using elementary mathematical methods and provide a new approach to the closure properties of irrational numbers. Further, we also study the closure properties of transcendental numbers.

KEYWORDS
Irrational numbers; Transcendental numbers; Dedekind cuts; Algebraic numbers

1. INTRODUCTION

In article,[1] a problem is posted to prove or disprove the following:

(a) The addition of two transcendental/irrational numbers is transcendental/ irrational
(b) The product of two transcendental /irrational numbers is transcendental/ irrational.

It is well known that the irrational numbers are those which cannot be represented as a ratio of integers. There are many popular irrational numbers like $\sqrt{2}$, Euler number e, Euler Mascheroni constant γ, golden ratio φ and the irrational π. Algebraic numbers are those numbers, which are roots of polynomials with integral coefficients, otherwise are transcendental numbers. Commonly known examples of transcendental numbers are e and π.

In this study, first, we account on the existence of irrational numbers and discuss the decimal representation of these numbers. In section 2 of this article, motivated from paper,[2] we establish the closure property of irrational numbers in a new way by using the infinite non-repeating and non-recurring decimal representation of these numbers. In section 3, the construction of transcendental numbers by Cantor's argument is discussed. Further, the main result on transcendental numbers which states that "The sum and product of transcendental numbers are not always transcendental, moreover at least one of the product or sum is transcendental," is proved.

2. CLOSURE PROPERTIES OF IRRATIONAL NUMBERS

It is quite reasonable to understand, how mathematicians came to know about the existence of irrational numbers. The first proof of the existence of irrational numbers is usually attributed to a Pythagorean who observed that in a right-angled triangle, with height and base 1 unit gives the hypotenuse side $\sqrt{2}$ unit. The more rigorous proof of construction of irrational numbers is given by mathematician Richard Dedekind which is popularly known as Dedekind cuts.[3]

The following are some known preliminary results involving irrational numbers.

LEMMA 1. The decimal part of irrational numbers is non-repeating and non-recurring.

PROOF. Consider a real number δ with repeating decimal

$$\delta = a_0 \cdot a_1 a_1 a_1 \ldots. \qquad \textbf{Equation 1.}$$

In order to prove that δ is a rational number, multiplying **Equation 1** by 10 and subtract it from **Equation 1**, we get

$$\delta = \frac{a_0(a_1 - 1)}{9},$$

which is clearly a rational number. In general, let ε be a real number with repeating digits having period n. Then,

$$\varepsilon = b_0.b_1b_2b_3 \dots b_nb_1b_2b_3 \dots b_nb_1b_2b_3 \dots. \qquad \textbf{Equation 2.}$$

Multiplying **Equation 2** by 10^n and subtracting it from **Equation 2**, we get

$$\varepsilon = \frac{b_0(b_1b_2b_3 \dots b_n - 1)}{(10^n - 1)},$$

which is also a rational number. This completes the proof.

We now investigate that why sum and product of two irrational numbers are rational or irrational. Though it is quite trivial by considering some examples like "Sum of $(\sqrt{2} + 1)$ and $(3 - \sqrt{2})$ and product of $(\sqrt{2} - 1)$ and $(\sqrt{2} + 1)$ are rational" and "Sum of $\sqrt{2}$ and $\sqrt{3}$ and product of $\sqrt{2}$ and $\sqrt{3}$ are irrational". But, we take a close look what is actually happening in these operations. For example, summing two irrational numbers $(\sqrt{2} + 1)$ and $(3 - \sqrt{2})$, one observes that the approximate sum of their decimal parts gives, 0.41421356237 + 0.58578643763 = 1.0 and terminating the fractional part, giving the overall sum is nothing but an integer. Now considering the sum of $\sqrt{2}$ and $\sqrt{3}$, in which the approximate sum fractional part gives 1.14626436994. In this example, the approximate sum of the fractional part is non-terminating and non-recurring even when number of decimal part increased, this is a property of irrational numbers.

From above two examples, we observe that the decimal part decides the rationality and irrationality of a number. We use this idea in order to investigate the problem posted.

The following identity is useful while proving the subsequent results.

PROPOSITION 1. Any irrational number α can be written as,

$$\alpha = [\alpha] + \{\alpha\},$$

where [ ] denotes the integral part and { } denotes the fractional part. Moreover, $\{\alpha\} = lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{10^i}$,

where $a_i \in 0,1,2 \dots 9$.

The following result shows that the sum of two irrational numbers is not always irrational.

THEOREM 1. The sum of two irrational numbers is not always irrational. In fact, the decimal part of the sum decides rationality or irrationality.

PROOF. Let γ and β are two irrational numbers. By virtue of **Proposition 1**,

$$\gamma = [\gamma] + \{\gamma\} \text{ and } \beta = [\beta] + \{\beta\}.$$

Moreover, $\{\gamma\} = lim_{n\to\infty} \sum_{i=1}^{n} \frac{c_i}{10^i}$ and $\{\beta\} = lim_{n\to\infty} \sum_{i=1}^{n} \frac{b_i}{10^i}$, where c_i and b_i are one of the digits 0 to 9. This implies that

$$\gamma + \beta = [\gamma] + \{\gamma\} + [\beta] + \{\beta\}.$$

Since, $[\gamma]$ and $[\beta]$ are integers then, their sum is also an integer. Now we take a look at sum of fractional parts, *i.e.*

$$\{\beta\} + \{\gamma\} = lim_{n\to\infty} \sum_{i=1}^{n} \frac{c_i + b_i}{10^i}.$$

The possible cases are

CASE 1. If $lim_{n\to\infty}\sum_{i=1}^{n}\frac{c_i+b_i}{10^i}=k_0.k_1k_1k_1....$, where k_i is one of the digits of 0 to 9. Then, $\gamma+\beta$ is rational number due to repeating property of decimal part.

CASE 2. If $lim_{n\to\infty}\sum_{i=1}^{n}\frac{c_i+b_i}{10^i}=k_0.k_1k_2k_3\ldots k_nk_1k_2k_3\ldots k_n\ldots k_1k_2....$, where k_i is one of the digits of 0 to 9 .Then, $\gamma+\beta$ is rational number due to recurring property of decimal part.

CASE 3. If $lim_{n\to\infty}\sum_{i=1}^{n}\frac{c_i+b_i}{10^i}=k_0.k_1k_2k_3k_4k_5\ldots$, where k_i is one of the digits of 0 to 9. Then, $\gamma+\beta$ is irrational number due to non-repeating and non-recurring property of decimal part.

Similarly, the product of two irrational numbers may not be irrational. The following result shows this fact.

THEOREM 2. The product of two irrational numbers is not always irrational. In fact, the decimal part of the product operation decides rationality or irrationality.

PROOF. Let γ and β are two irrational numbers. By virtue of **Proposition 1**,

$$\gamma\times\beta=([\gamma]+\{\gamma\})\times([\beta]+\{\beta\})$$

$$=[\gamma]\times[\beta]+[\gamma]\times\{\beta\}+\{\gamma\}\times[\beta]+\{\gamma\}\times\{\beta\}.$$

The affecting property of irrationality is decimal part, now we concentrate at decimal part of product of γ and β, since $[\gamma]\times[\beta]$ is integer.

Since, $[\gamma]\times\{\beta\}+\{\gamma\}\times[\beta]+\{\gamma\}\times\{\beta\}$ is $lim_{n\to\infty}\sum_{i,j=1}^{n}\frac{[\beta]c_i+[\gamma]b_i+c_ib_j}{10^i}$, possible cases are

CASE 1. If $lim_{n\to\infty}\sum_{i,j=1}^{n}\frac{[\beta]c_i+[\gamma]b_i+c_ib_j}{10^i}=k_0.k_1k_1k_1\ldots$, where k_i is one of the digits of 0 to 9. Then, $\gamma\times\beta$ is rational number due to repeating property of decimal part.

CASE 2. If $lim_{n\to\infty}\sum_{i,j=1}^{n}\frac{[\beta]c_i+[\gamma]b_i+c_ib_j}{10^i}=k_0.k_1k_2k_3\ldots k_nk_1k_2k_3\ldots k_nk_1k_2\ldots$, where k_i is one of the digits of 0 to 9. Then, $\gamma\times\beta$ is rational number due to recurring property of decimal part.

CASE 3. If $lim_{n\to\infty}\sum_{i,j=1}^{n}\frac{[\beta]c_i+[\gamma]b_i+c_ib_j}{10^i}=k_0.k_1k_2k_3k_4k_5\ldots$ where $k_i$is one of the digits of 0 to 9. Then, $\gamma\times\beta$ is irrational number due to non-repeating and non-recurring property of decimal part.

3. CLOSURE PROPERTIES OF TRANSCENDENTAL NUMBERS

Joseph Liouville first proved the existence of transcendental numbers. He gave the first decimal examples such as the Liouville constant i.e. $\sum_{k=1}^{\infty}10^{-k!}$. The construction of transcendental numbers also comes to know from Cantor arguments. By definition, a set A is said to be countable if there exists a one to one correspondence between A to **N**, where **N** is set of natural numbers. It is well known that the set of real numbers are uncountable by Cantor diagonalisation argument. Since, Cantor proved, in the set of real numbers, the numbers which are roots of polynomials with integral coefficients i.e. the set of algebraic numbers is countable, so there also exist numbers which are not roots of polynomial, uncountable i.e. the set of transcendental numbers.[4]

The following result shows that for any two transcendental numbers, at least one of sum or product of these numbers is transcendental.

THEOREM 3. If ω and φ are two transcendental numbers, then at least one of $\omega+\varphi$ or $\omega\times\varphi$ is transcendental.

PROOF. We use method of contradiction to prove this result. Consider two transcendental numbers ω and φ. Suppose if possible $\omega+\varphi$ and $\omega\times\varphi$ both are algebraic numbers.

Then,

$$\omega + \varphi = a_0 \text{ and } \omega \times \varphi = a_1, \qquad \textbf{Equation 3.}$$

where a_0 and a_1 are algebraic. Simplification of **Equation 3** gives a polynomial with algebraic coefficients, $\omega^2 + a_1\omega - a_0 = 0$. Since, polynomial with algebraic coefficients always gives algebraic roots,[5] which contradicts the fact that ω is a transcendental root. This completes the proof.

CONCLUSION

In this article, we see the existence and construction of irrational and transcendental numbers. Followed by it, by using the concepts of decimal representation, we get the conclusion that the sum and product of irrational and transcendental numbers are not always irrational or transcendental.

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ABOUT THE STUDENT AUTHOR

Shekh Mohammed Zahid is a first-year undergraduate student of BSc(Hons) in Mathematics and Computing in the Institute of Mathematics and Applications, Odisha, India. His research interest includes number theory and theoretical physics. He likes to play chess.

PRESS SUMMARY

In this article, we provide a solution to a problem posted on American Journal of Undergraduate Research (AJUR) regarding closure properties of irrational and transcendental numbers under the operation of addition and multiplication. Though the problem already has a solution, in this article, we provide a new approach to solving it.